Analytic Combinatorics Exercise Sheet 1

Exercises for the session on 20/3/2017

Problem 1.1

Show
$$[z^n]\left(\frac{1-\sqrt{1-4z}}{2}\right) = \frac{1}{n}\binom{2n-2}{n-1}$$
(use the generalised binomial theorem $(1+x)^\alpha = \sum_{k\geq 0}\binom{\alpha}{k}x^k$, where
$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad)$$
and derive an asymptotic formula for
$$\frac{1}{n}\binom{2n-2}{n-1}$$
(use Stirling's formula $n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$).

Problem 1.2

This question concerns the number of ways a string of n identical letters, say x, can be 'bracketed'. The rule is best stated recursively: x itself is a bracketing and if $\sigma_1, \sigma_2, \ldots, \sigma_k$ with $k \geq 2$ are bracketed expressions, then the k-ary product $(\sigma_1\sigma_2\cdots\sigma_k)$ is a bracketing. For instance: $(((xx)x(xxx))((xx)(xx)x))$.

Let $\mathcal{S}$ denote the class of all bracketings, where size is taken to be the number of instances of x, and let $S(z)$ denote the ordinary generating function for $\mathcal{S}$.

Show
$$S(z) = \frac{1}{4}(1+z-\sqrt{1-6z+z^2})$$
(remember to justify why we choose the negative root here).

Problem 1.3

Let $T_n^{\{0,r\}}$ denote the number of rooted plane r-ary trees on n vertices. Find $T_n^{\{0,r\}}$ (use Lagrange's Inversion Theorem) and show

$$T_{2n+1}^{\{0,2\}} \sim \frac{4^n}{\sqrt{\pi n^3}}.$$

Problem 1.4

Let $A(z) = \sum_{n\geq 0} A_n z^n$ denote the ordinary generating function for the Fibonacci numbers (defined by $A_0 = A_1 = 1$ and $A_{n+2} = A_{n+1} + A_n$ for $n \geq 0$). Show

$$A(z) = \frac{1}{1 - z - z^2}$$

and hence show

$$A_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1}$$

(use partial fractions).

Problem 1.5

Let $A_n = \alpha n + \beta$ for all $n \geq 0$. Find the ordinary generating function $A(z) = \sum_{n\geq 0} A_n z^n$.

Problem 1.6

Find the ordinary generating function $A(z) = \sum_{n\geq 0} A_n z^n$, where A_n denotes the number of integers between 0 and $10^m - 1$ whose digits sum to n.